RESONANT DECOMPOSITIONS AND THE I-METHOD FOR CUBIC NONLINEAR SCHRÖDINGER ON \mathbb{R}^2

J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO

ABSTRACT. The initial value problem for the cubic defocusing nonlinear Schrödinger equation $i\partial_t u + \Delta u = |u|^2 u$ on the plane is shown to be globally well-posed for initial data in $H^s(\mathbb{R}^2)$ provided s>1/2. The proof relies upon an almost conserved quantity constructed using multilinear correction terms. The main new difficulty is to control the contribution of resonant interactions to these correction terms. The resonant interactions are significant due to the multidimensional setting of the problem and some orthogonality issues which arise.

1. Introduction

We consider the Cauchy problem for the cubic defocusing nonlinear Schrödinger (NLS) equation

(1.1)
$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u, \\ u(0,x) = u_0(x) \in H_x^s(\mathbb{R}^2), \end{cases}$$

in a Sobolev space $H^s_x(\mathbb{R}^2)$, where the unknown function $u: J \times \mathbb{R}^2 \longrightarrow \mathbb{C}$ is a strong solution to (1.1) on a time interval $J \subset \mathbb{R}$ in the sense that $u \in C^0_{t,\text{loc}}H^s_x(\mathbb{R}^d)$ and u obeys the integral equation

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-t')\Delta}[|u|^2u(t')] dt'$$

for $t \in J$. Here of course the propagators $e^{it\Delta}$ are defined via the Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) \ dx$$

by the formula

$$\widehat{e^{it\Delta}f}(\xi) := e^{-it|\xi|^2} \hat{f}(\xi)$$

and the Sobolev space $H^s_x(\mathbb{R}^2)$ is similarly defined via the Fourier transform using the norm

$$||f||_{H_x^s(\mathbb{R}^2)} := ||\langle \xi \rangle^s \hat{f}(\xi)||_{L_\xi^2(\mathbb{R}^2)}$$

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where $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$. For later use we shall also need the homogeneous Sobolev norms

$$||f||_{\dot{H}_{x}^{s}(\mathbb{R}^{2})} := |||\xi|^{s} \hat{f}(\xi)||_{L_{\varepsilon}^{2}(\mathbb{R}^{2})}.$$

We are interested primarily in the global-in-time problem, in which we allow J to be the whole real line \mathbb{R} .

Both the local and global-in-time Cauchy problems for this NLS equation (1.1) have attracted a substantial literature [30], [10], [22], [20] [26], [5], [6], [17], [25], [9], [3], [11]. The local well-posedness theory is now well understood; in particular, one has local well-posedness in $H_x^s(\mathbb{R}^2)$ for all $s \geq 0$, and if s is strictly positive then a solution can be continued unless the $H_x^s(\mathbb{R}^2)$ norm of the solution goes to infinity at the blowup time (see e.g. [8], [27]). Also, due to the smooth nature of the nonlinearity, any local $H^s(\mathbb{R}^2)$ solution can be expressed as the limit (in $C_{t,\text{loc}}^0H_x^s$) of smooth solutions. For s < 0 the solution map ceases to be uniformly continuous [11] and may possibly even be undefined, though it is known that well-posedness can be recovered for other spaces rougher than $L_x^2(\mathbb{R}^2)$ [25], [9]. The space $L_x^2(\mathbb{R}^2)$ is the critical space for this equation, as it is invariant under the scaling symmetry

$$(1.2) u(t,x) \mapsto \frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$$

of (1.1).

Now we turn attention to the global-in-time well-posedness problem. Based on the local well-posedness theory, standard limiting arguments, and the time reversal symmetry $u(t,x) \mapsto \overline{u(-t,x)}$, global well-posedness of (1.1) for arbitrarily large data¹ in $H_x^s(\mathbb{R}^2)$ for some s > 0 follows if an a priori bound of the form

$$(1.3) ||u(T)||_{H_x^s(\mathbb{R}^2)} \le C(s, ||u_0||_{H_x^s(\mathbb{R}^2)}, T)$$

can be established for all times $0 < T < \infty$ and all smooth-in-time, Schwartz-in-space solutions $u: [0,T] \times \mathbb{R}^2 \to \mathbb{C}$, where the right-hand side is some finite quantity depending only upon s, $\|u_0\|_{H^s_x(\mathbb{R}^2)}$, and T. Thus we shall henceforth restrict attention to such smooth solutions, which will in particular allow us to justify all formal computations, such as verification of conservation laws.

As is well known, the equation NLS enjoys two useful conservation laws, the energy conservation law

(1.4)
$$E(u(t)) := \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(t, x)|^4 dx = E(u_0).$$

and the mass conservation law

$$||u(t)||_{L^2(\mathbb{R}^2)} = ||u_0||_{L^2(\mathbb{R}^2)}.$$

From these laws one easily establishes (1.3) for s = 1 (with bounds uniform in T), and with some additional arguments one can then deduce the same claim for s > 1 (with the best known bounds growing polynomially in T; see [26], [12]). The mass

¹Global well-posedness and even scattering is known when the mass $\|u_0\|_{L^2_x(\mathbb{R}^2)}$ is sufficiently small (see e.g. [8], [27]), or if suitable decay conditions (e.g. $xu_0 \in L^2_x(\mathbb{R}^2)$ are also imposed on the initial data [30]). Our interest here however is in the large data case with no further decay conditions beyond the requirement that u_0 lie in $H^s_x(\mathbb{R}^2)$.

conservation law (1.5) also gives (1.3) for s = 0, but unfortunately this does not immediately imply any result for s > 0 except in the small mass case².

It is conjectured that the equation (1.1) is globally well-posed in $H_x^s(\mathbb{R}^2)$ for all $s \geq 0$, and in particular (1.3) holds for all s > 0. This conjecture remains open (though in the radial case, the higher dimensional analogue of this conjecture has recently been settled in [29]). However, there has been some progress in improving the $s \geq 1$ results mentioned earlier. The first breakthrough was by Bourgain [5], [6], who established (1.3) (and hence global well-posedness in $H_x^s(\mathbb{R}^2)$) for all s > 3/5, using what is now referred to as the Fourier truncation method.

In [17] the bound (1.3) was established for all s > 4/7, using the "I-method" developed by the authors in [14], [15] (see also [23]). The main result of this paper is the following improvement:

Theorem 1.1 (Main theorem). The bound (1.3) holds for all s > 1/2. In particular, the Cauchy problem (1.1) is globally well-posed in $H_x^s(\mathbb{R}^2)$ for all s > 1/2.

Our arguments refine our previous analysis in [17] by adding a "correction term" to a certain modified energy functional E(Iu), as in [15] or [16], in order to damp out some oscillations in that functional; also, we establish some more refined estimates on the multilinear symbols appearing in those integrals. The main new difficulty is that, due to the non-integrability and multidimensional setting of this equation (in contrast³ to [15]), the direct analogue of the correction terms used in [15], [16] contains a singular symbol and is thus intractable to estimate. We get around this new difficulty by truncating the correction term to non-resonant interactions, and dealing with the resonant interactions separately by some advanced estimates of $X^{s,b}$ type. This method seems quite general and should lead to improvements in global well-posedness results for other non-integrable evolution equations which are currently obtained by the "first-generation" I-method (i.e. without correction terms). A resonant decomposition similar to that employed here appeared previously in the work [7], and more recently in [1].

Inserting the above theorem into the results of [4] (which employ the pseudo-conformal transform) we conclude that the equation (1.1) is globally well-posed with scattering when the initial data obeys $\langle x \rangle^s u_0 \in L^2_x(\mathbb{R}^2)$ for any s > 1/2.

During the preparation of this manuscript, we learned that Fang and Grillakis [21] had also obtained Theorem 1.1, in fact for $s \ge 1/2$, by a different method based upon a new type of Morawetz inequality. The Fang-Grillakis interaction Morawetz estimate has recently [13] been improved and combined with the *I*-method (following the general scheme from [18]) to prove that (1.1) is globally well-posed in H^s for s > 2/5. The techniques leading to the improved energy increment control obtained in this paper (see (2.3) which is $N^{-1/2}$ better than what was obtained in [17] and used in [18], [13]) may also improve the "almost Morawetz" increment in [13] by $N^{-1/2}$. Such an improvement combined with (2.3) would improve the global well-posedness result to s > 4/13. The arguments in [21], [13] are based on Morawetz inequalities and are thus restricted to the defocusing case. Provided the

 $^{^2 \}text{In order to establish a global well-posedness result in } L^2_x(\mathbb{R}^2),$ it is instead necessary to obtain an a priori spacetime bound such as $\|u\|_{L^4_{t,x}([0,T]\times\mathbb{R}^2)} \leq C(\|u_0\|_{L^2_x(\mathbb{R}^2)}).$ See [24], [4], [28] for

³The equation considered in [16] was also non-integrable, but because it was one-dimensional there was still enough cancellation to prevent the contribution of the resonant interactions from becoming singular.

mass of the initial data is less than the mass of the ground state, Theorem 1.1 also holds true for the focusing analog of (1.1) (see Remark 2.4 below). The focusing problem is expected to be globally well-posed for L^2 initial data with mass less than the ground state mass.

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2. Setting up the I-method

We now begin the proof of Theorem 1.1. As in all other applications of the I-method, we will reduce matters to one of constructing a certain modified energy functional $\tilde{E}(u(t))$ and demonstrating that it has certain almost conservation properties.

By the discussion in the introduction, it suffices to prove (1.3) in the range 1/2 < s < 1. Henceforth we fix s. We adopt the usual notation that $X \lesssim Y$ or $Y \gtrsim X$ denotes an estimate of the form $X \leq C(s)Y$, for some constant $0 < C(s) < \infty$ depending only on s. We also write $X \sim Y$ for $X \lesssim Y \lesssim X$, and X = O(Y) for $|X| \lesssim Y$.

We will use exponents a+ and a- to denote $a+\varepsilon$ and $a-\varepsilon$ for arbitrarily small exponents $\varepsilon>0$, and allow the implied constants in the \lesssim notation to depend on ε . Thus for instance if we write $X\lesssim N^{1+}Y$, this means that for every ε there exists a constant $C(s,\varepsilon)$ such that $X\leq C(s,\varepsilon)N^{1+\varepsilon}Y$.

Let $N \gg 1$ be a large parameter to be chosen later (it will eventually depend on T, s, and the size of the initial data u_0). We define the Fourier multiplier $I = I_N$ by

$$\widehat{Iu}(\xi) := m(\xi)\hat{u}(\xi)$$

where m is a smooth non-negative radial symbol which equals 1 when $|\xi| \leq N$, equals $(|\xi|/N)^{s-1}$ for $|\xi| \geq 2N$, and smoothly interpolates between the two in the region $N \leq |\xi| \leq 2N$. We shall abuse notation and write $m(|\xi|)$ for $m(\xi)$, thus for instance m(N) = 1.

The "first-generation" I-method revolves around the modified energy

$$(2.1) \quad E(Iu(t)) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla Iu(t,x)|^2 + \frac{1}{4} |Iu(t,x)|^4 \ dx \sim ||Iu||_{\dot{H}^1_x(\mathbb{R}^2)}^2 + ||Iu||_{L^4_x(\mathbb{R}^2)}^4,$$

and in particular establishing an almost conservation law for this quantity. Here, we shall introduce a slight variant $\tilde{E}(u(t))$ of E(Iu(t)) and establish an almost conservation law for that quantity instead. More precisely, we shall show the following:

Theorem 2.1 (Existence of an almost conserved quantity). There exists a functional $\tilde{E} = \tilde{E}_N : \mathcal{S}_x(\mathbb{R}^2) \to \mathbb{R}$ defined on Schwartz functions $u \in \mathcal{S}_x(\mathbb{R}^2)$ with the following properties.

• (Fixed-time bounds) For any $u \in \mathcal{S}_x(\mathbb{R}^2)$, we have

$$|E(Iu) - \tilde{E}(u)| \lesssim N^{-1+} ||Iu||_{H_x^1(\mathbb{R}^2)}^4.$$

• (Almost conservation law) If $||u_0||_{L^2_x(\mathbb{R}^2)} \leq A$ and $E(Iu_0) \leq 1$, and u is a smooth-in-time, Schwartz-in-space solution to (1.1) on a time interval $[0,t_0]$, then if t_0 is sufficiently small depending on A, we have

(2.3)
$$|\tilde{E}(u(t_0)) - \tilde{E}(u_0)| \lesssim C(A)N^{-2+}$$

for some constant C(A) depending only on A.

Remark 2.2. The precise value of the exponent -1+ in (2.2) is not particularly important; any negative exponent would have sufficed here. However, the exponent -2+ in (2.3) is directly tied to the restriction s>1/2 in our main theorem. More generally, an exponent of $-\alpha+$ in this almost conservation law translates to a constraint $s>2/(2+\alpha)$. In [17], the first-generation modified energy E(Iu) was shown to obey an almost conservation law with $\alpha=3/2$, which ultimately led to the constraint s>4/7. Note that in order to get arbitrarily close to the scaling exponent s=0, one would need α to be arbitrarily large, which looks unlikely to be achieved with this method due to the lack of complete integrability.

We shall prove Theorem 2.1 in later sections. For the remainder of this section, we show how Theorem 2.1 implies Theorem 1.1.

Proof of Theorem 1.1 assuming Theorem 2.1. Fix u, u_0 , T as in Theorem 1.1, and write $A := 1 + \|u_0\|_{H_x^s(\mathbb{R}^d)}$. We let $\lambda \geq 1$ be a scaling parameter to be chosen shortly, and define the rescaled solution $u^{(\lambda)} : [0, \lambda^2 T] \times \mathbb{R}^2 \to \mathbb{C}$ as per (1.2), thus

$$u^{(\lambda)}(t,x) := \frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}).$$

Now let $N \gg 1$ also be a parameter to be chosen later (it will depend on T and A). A simple computation (see equation (3.10) of [17]) shows that

$$E(Iu^{(\lambda)}(0)) \lesssim N^{2-2s} \lambda^{-2s} A^4.$$

Thus we can arrange

$$(2.4) E(Iu^{(\lambda)}(0)) \le 1/3$$

by choosing

$$\lambda := C(s, A)N^{(1-s)/s}$$

for a suitable quantity C(s, A). Also, from mass conservation (and scale-invariance) we also know that

(2.6)
$$||Iu^{(\lambda)}(t)||_{L_x^2(\mathbb{R}^2)} \le A.$$

From (2.2), (2.1) we conclude that

$$(2.7) |E(Iu^{(\lambda)}(t)) - \tilde{E}(u^{(\lambda)}(t))| \lesssim N^{-1+} (A^4 + E(Iu^{(\lambda)}(t))^2).$$

We now claim that (for ε chosen suitably small, and for N chosen suitably large)

(2.8)
$$E(Iu^{(\lambda)}(\lambda^2 T)) \le 2/3.$$

To see this, suppose for contradiction that this were not the case; then there exists $0 < T' < \lambda^2 T$ such that

(2.9)
$$E(Iu^{(\lambda)}(T')) = 2/3$$

but that

$$E(Iu^{(\lambda)}(t)) \le 2/3$$
 for all $0 \le t \le T'$.

Applying (2.7) we conclude (if N is sufficiently large depending on A) that

$$\tilde{E}(u^{(\lambda)}(t)) \leq 1$$
 for all $0 \leq t \leq T'$.

Applying (2.3) repeatedly (and exploiting time translation invariance), we conclude that

$$|\tilde{E}(u^{(\lambda)}(T')) - \tilde{E}(u^{(\lambda)}(0))| \lesssim C(A)N^{-2+}T' \leq C(A)N^{-2+}\lambda^2T$$

and hence by (2.7)

$$|E(Iu^{(\lambda)}(T')) - E(Iu^{(\lambda)}(0))| \le C(A)N^{-2+}\lambda^2T + N^{-1+}A^4.$$

From (2.5) and the hypothesis s > 1/2, we see that the net powers of N on the right are negative. Thus we can choose N so large (depending on A, T) that

$$|E(Iu^{(\lambda)}(T')) - E(Iu^{(\lambda)}(0))| < 1/3.$$

But this contradicts (2.4), (2.9). Thus (2.8) must hold. From this, (2.6), and some Fourier analysis we deduce

$$||u^{(\lambda)}||_{\dot{H}^{s}_{x}(\mathbb{R}^{2})} \lesssim A +$$

and hence (on undoing the scaling)

$$||u||_{\dot{H}^{s}_{\alpha}(\mathbb{R}^{2})} \lesssim A\lambda^{s}$$

which (together with mass conservation) gives (1.3) as desired.

Remark 2.3. By pursuing the above analysis more carefully, we in fact obtain a bound of the form

$$||u(T)||_{H_x^s(\mathbb{R}^2)} \lesssim (1 + ||u_0||_{H_x^s(\mathbb{R}^2)})^{C_s} (1 + T)^{\frac{s(1-s)}{2(2s-1)} + \frac{s(1-s)}{2(2s-1)}}$$

for some $C_s > 0$.

Remark 2.4. Theorems 2.1 and 1.1 also hold for the focusing analog of (1.1) (replacing $|u|^2u$ by $-|u|^2u$) provided we also assume $||u_0||_{L^2_x(\mathbb{R}^2)} < ||Q||_{L^2_x(\mathbb{R}^2)}$. Here Q is the ground state profile which arises as the unique (up to translations) positive solution of $-Q + \Delta Q = -Q^3$. Indeed, most of the argument remains unchanged (particularly those involving the local theory, or multilinear estimates). The only new difficulty arises when trying to use the energy E(u) to control the kinetic component $||u||^2_{H^1(\mathbb{R}^2)}$, since the potential energy component of the energy is now negative. However, the sharp Gagliardo-Nirenberg inequality [31] allows one to achieve this control (losing a constant, of course) provided that $||u||_{L^2_x(\mathbb{R}^2)} < ||Q||_{L^2_x(\mathbb{R}^2)}$, allowing one to continue the argument without difficulty. As the modifications are rather standard we do not detail them further here.

It remains to prove Theorem 2.1. There are clearly three components to this task: firstly, to construct the functional \tilde{E} ; secondly, to establish the fixed-time bound (2.2); and thirdly, to obtain the almost conservation law (2.3). The first two tasks are straightforward and will be accomplished in the next two sections. The third is substantially more difficult and will occupy the remainder of the paper.

3. Construction of the modified energy functional

We begin with the construction of the modified energy functional \tilde{E} . As in previous literature on the *I*-method (e.g. [15], [16], [17]), it is convenient to introduce some notation for multilinear expressions involving u.

Let k be an integer, let $\Sigma_k \subset (\mathbb{R}^2)^k$ denote the space

$$\Sigma_k := \{ (\xi_1, \dots, \xi_k) \in (\mathbb{R}^2)^k : \xi_1 + \dots + \xi_k = 0 \},$$

with the measure induced from Lebesgue measure $d\xi_1 \dots d\xi_{k-1}$ by pushing forward under the map

$$(\xi_1,\ldots,\xi_{k-1})\mapsto (\xi_1,\ldots,\xi_{k-1},-\xi_1-\ldots-\xi_{k-1}).$$

If $M: \Sigma_k \to \mathbb{C}$ is a smooth tempered symbol, and $u_1, \ldots, u_k \in \mathcal{S}(\mathbb{R}^2)$ are Schwartz functions, we define the k-linear functional

$$\Lambda_k(M; u_1, \dots, u_k) := \frac{1}{(2\pi)^{2(k-1)}} \operatorname{Re} \int_{\Sigma_k} M(\xi_1, \dots, \xi_k) \widehat{u_1}(\xi_1) \dots \widehat{u_k}(\xi_k).$$

When k is even, we abbreviate

$$\Lambda_k(M; u) := \Lambda_k(M; u, \overline{u}, \dots, u, \overline{u}).$$

We observe that the quantity $\Lambda_k(M; u)$ is invariant if one permutes the even arguments $\xi_2, \xi_4, \ldots, \xi_k$ of M, the odd arguments $\xi_1, \xi_3, \ldots, \xi_{k-1}$ of M, as well as the additional symmetry

$$M(\xi_1, \xi_2, \dots, \xi_{k-1}, \xi_k) \mapsto \overline{M}(\xi_2, \xi_1, \dots, \xi_k, \xi_{k-1})$$

which swaps the odd and even arguments, and also conjugates M. This generates a finite group G_k of order $|G_k| = (k/2)! \times (k/2)! \times 2$ of symmetries, acting on Σ_k and thus on the class m of symbols. This leads to the symmetrization rule

(3.1)
$$\Lambda_k(M; u) = \Lambda_k([M]_{\text{sym}}; u)$$

where $[M]_{\text{sym}} := \frac{1}{|G_k|} \sum_{g \in G_k} gM$ is the G_k -symmetric component of M. Using the above notation and the Fourier inversion formula, we observe that

$$E(Iu) = \Lambda_2(\sigma_2; u) + \Lambda_4(\sigma_4; u)$$

where

$$\sigma_2(\xi_1, \xi_2) := -\frac{1}{2}\xi_1 m_1 \cdot \xi_2 m_2 = \frac{1}{2}|\xi_1|^2 m_1^2$$

and

$$\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4) := \frac{1}{4} m_1 m_2 m_3 m_4$$

and we abbreviate $m(\xi_j)$ as m_j . Observe that σ_2 and σ_4 are both symmetric with respect to the group G_k .

Now we investigate the behaviour of these multilinear forms in time. If u is a smooth-in-time, Schwartz-in-space solution to (1.1), and M is independent of time and symmetric with respect to G_k , then from the identity

$$u_t = i\Delta u - iu\overline{u}u$$

arising from (1.1), together with some Fourier analysis, we have the differentiation formula

$$\partial_t \Lambda_k(M; u(t)) = \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}(ikX(M); u(t))$$

= $\Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}([ikX(M)]_{\text{sym}}; u(t))$

where α_k is the symbol

$$\alpha_k(\xi_1,\ldots,\xi_k) := -|\xi_1|^2 + |\xi_2|^2 - \ldots - |\xi_{k-1}|^2 + |\xi_k|^2$$

(in particular, we have $\alpha_2 = 0$ on Σ_2) and X(M) is the extended symbol

$$X(M)(\xi_1,\ldots,\xi_{k+2}) := M(\xi_{123},\xi_4,\ldots,\xi_{k+2})$$

where we use the notational convention $\xi_{ab} := \xi_a + \xi_b$, $\xi_{abc} := \xi_a + \xi_b + \xi_c$, etc. Note that $iM\alpha_k$ is already symmetric with respect to G_k and thus does not require further symmetrising.

As one particular instance of the above computations and the fundamental theorem of calculus, we have

$$E(Iu(t)) - E(Iu(0)) = \int_0^t \partial_t E(Iu(t')) dt'$$

$$= \int_0^t \Lambda_4([-2iX(\sigma_2)]_{\text{sym}} + i\sigma_4\alpha_4; u(t')) dt'$$

$$- \int_0^t \Lambda_6([4iX(\sigma_4)]_{\text{sym}}; u(t')) dt'.$$

In the case $m \equiv 1$ (which corresponds to s=1 or $N=\infty$), one easily computes that $[-2iX(\sigma_2)]_{\mathrm{sym}} + i\sigma_4\alpha_4$ and $[4iX(\sigma_4)]_{\mathrm{sym}}$ both vanish, thus giving a proof of energy conservation. When m is the multiplier from the previous section, these symbols do not vanish at high frequencies (when $\max(|\xi_1|,\ldots,|\xi_k|) \geq N$) but it turns out that the right-hand side can still be estimated by an expression which decays in N as $O(N^{-3/2+})$; see [17]. In fact, only the Λ_4 terms are as large as $O(N^{-3/2+})$; a closer inspection of the arguments in [17] show that the Λ_6 term is as least as small as $O(N^{-2+})$. The strategy is thus to modify the quantity E(Iu) so that the time derivative has less of a Λ_4 term and more of a Λ_6 term. Specifically, we shall define

(3.2)
$$\tilde{E}(u) := \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u)$$

for some G_4 -symmetric $\tilde{\sigma}_4$ to be chosen shortly. Computing as before we have

$$\tilde{E}(u(t)) - \tilde{E}(u(0)) = \int_0^t \Lambda_4([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha_4; u(t')) dt'$$
$$- \int_0^t \Lambda_6([4iX(\tilde{\sigma}_4)]_{\text{sym}}; u(t')) dt'.$$

An initial guess for $\tilde{\sigma}_4$ would thus be

$$\tilde{\sigma}_4 := \frac{[2iX(\sigma_2)]_{\text{sym}}}{i\alpha_4}.$$

However this choice runs into the problem that α_4 can vanish in the resonant interaction case when ξ_{12} and ξ_{14} are either zero or orthogonal. The first situation is easier to handle. In fact one can write

(3.3)
$$\alpha_4 := -2\xi_{12} \cdot \xi_{14} = -2|\xi_{12}||\xi_{14}|\cos\angle(\xi_{12}, \xi_{14})$$

and

$$(3.4) [2iX(\sigma_2)]_{\text{sym}} = \frac{1}{4}(-m_1^2|\xi_1|^2 + m_2^2|\xi_2|^2 - m_3^2|\xi_3|^2 + m_4^2|\xi_4|^2).$$

In particular, when all frequencies are less than N in magnitude, thus $\max_{1 \le i \le 4} |\xi_i| \le N$, then we have computed

$$(3.5) [2iX(\sigma_2)]_{\text{sym}} = \frac{1}{4}i\alpha_4$$

and so the vanishing of the denominator is cancelled by the numerator. A similar argument can be used when $\xi_{12}=0$ or $\xi_{14}=0$. Unfortunately, this cancellation

is lost when one has one or more high frequencies; this is in contrast to the onedimensional situation in [15], [17], where the resonant interactions are simpler (and in [15], one also has complete integrability to provide further cancellations).

Motivated by the above discussion, we shall in fact set

(3.6)
$$\tilde{\sigma}_4 := \frac{[2iX(\sigma_2)]_{\text{sym}}}{i\alpha_4} 1_{\Omega_{nr}}$$

where $1_{\Omega_{nr}}$ is the indicator function to the non-resonant set (3.7)

$$\Omega_{nr} := \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 : \max_{1 \le j \le 4} |\xi_j| \le N \} \cup \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 : |\cos \angle (\xi_{12}, \xi_{14})| \ge \theta_0 \},$$

where $0 < \theta_0 < 1/100$ is a parameter to be chosen later (we will shortly take $\theta_0 := 1/N$). Note that while the angle $\angle(\xi_{12}, \xi_{14})$ is undefined when ξ_{12} or ξ_{14} vanishes, but this set has measure zero and can be ignored.

Remark 3.1. The presence of the expression $|\cos \angle(\xi_{12}, \xi_{14})|$ is the key to all of our improvements over the previous work in [17]. However, as this expression involves three of the four frequencies in Σ_4 , exploiting this term properly will turn out to be a significant technical headache, requiring many decompositions of the frequency variables to handle.

We now define \tilde{E} by (3.2) with $\tilde{\sigma}_4$ as in (3.6). To prove Theorem 2.1, it thus suffices to prove the following two propositions.

Proposition 3.2 (Fixed-time estimate). Let the notation be as above. Then for any $u \in \mathcal{S}_x(\mathbb{R}^2)$, we have

$$(3.8) |\Lambda_4(\sigma_4 - \tilde{\sigma}_4; u)| \lesssim \theta_0^{-1} N^{-2+} ||Iu||_{H_x^1(\mathbb{R}^2)}^4.$$

Proposition 3.3 (Almost conservation law). Let the notation be as above. If $||u_0||_{L_x^2(\mathbb{R}^2)} \leq A$ and $E(Iu_0) \leq 1$, and u is a smooth-in-time, Schwartz-in-space solution to (1.1) on a time interval $[0,t_0]$, then if t_0 is sufficiently small depending on A, we have

$$\left| \int_{0}^{t_0} \Lambda_4([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha_4; u(t)) \ dt \right| \lesssim C(A)[N^{-2+} + \theta_0^{1/2}N^{-3/2+} + \theta_0^{-1}N^{-3+}]$$

$$\left| \int_0^{t_0} \Lambda_6([4iX(\tilde{\sigma}_4)]_{\text{sym}}; u(t)) \ dt \right| \lesssim C(A)[N^{-2+} + \theta_0^{1/2}N^{-3/2+} + \theta_0^{-1}N^{-3+}].$$

Indeed, by setting $\theta_0 := 1/N$ we obtain the desired result. In fact, we will see below (see Remark 5.4 below and the two propositions preceding it) that the 6-linear estimate degenerates with growing θ_0 while the 4-linear estimate improves with θ_0 and that the choice $\theta_0 = 1/N$ puts these contributions to the energy increment in balance.

The rest of the paper is now devoted to the proof of these two propositions.

4. The fixed time estimate

In this section we prove Proposition 3.2, which is in fact rather easy. From Plancherel's theorem, it suffices to show that

$$\int_{\Sigma_4} |\sigma_4(\xi) - \tilde{\sigma}_4(\xi)| \prod_{i=1}^4 \frac{|\hat{u}_j(\xi_j)|}{m_j \langle \xi_j \rangle} \lesssim \theta_0^{-1} N^{-2+} \prod_{i=1}^4 ||u_i||_{L^2(\mathbb{R}^2)}$$

for any $u_1, u_2, u_3, u_4 \in L^2(\mathbb{R}^2)$, where $\xi := (\xi_1, \xi_2, \xi_3, \xi_4)$.

From (3.5), (3.6), (3.7) we know that $\sigma_4(\xi) - \tilde{\sigma}_4(\xi)$ vanishes when $\max_{1 \leq j \leq 4} |\xi_j| \leq N$, so we may restrict to the region $\max_{1 \leq j \leq 4} |\xi_j| > N$. We now need the following bound.

Lemma 4.1. For any $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4$, We have

$$|[2iX(\sigma_2)]_{\text{sym}}| \lesssim \min(m_1, m_2, m_3, m_4)^2 |\xi_{12}| |\xi_{14}|.$$

Proof. Let $f(\xi) := m(\xi)^2 |\xi|^2$. In light of (3.4), it suffices to show that

$$|f(\xi_1) - f(\xi_2) + f(\xi_3) - f(\xi_4)| \lesssim \min(m_1, m_2, m_3, m_4)^2 |\xi_{12}| |\xi_{14}|.$$

Using symmetries, we may assume that $|\xi_1| \ge |\xi_2|, |\xi_3|, |\xi_4|$ and $|\xi_{12}| \ge |\xi_{14}|$. In particular $\min(m_1, m_2, m_3, m_4) = m_1$.

First assume that $|\xi_{12}|, |\xi_{14}| \gtrsim |\xi_1|$. Then we can estimate all four terms on the left-hand side by $O(m_1^2|\xi_1|^2)$, and the claim follows.

Now assume that $|\xi_{12}| \sim |\xi_1|$ but that $|\xi_{14}| \ll |\xi_1|$. We write the left-hand side as

$$|(f(\xi_1) - f(\xi_1 - f(\xi_{14}))) + (f(\xi_3) - f(\xi_3 + \xi_{14}))|.$$

Note that $\nabla f(\xi) = O(m(\xi)^2 |\xi|)$, and $m(\xi)^2 |\xi|$ is an increasing function of $|\xi|$, so by the fundamental theorem of calculus we have

$$|f(\xi_1) - f(\xi_1 - f(\xi_{14}))|, |f(\xi_3) - f(\xi_3 + \xi_{14})| \lesssim m_1^2 |\xi_1| |\xi_{14}|$$

and the claim follows.

Finally, suppose that $|\xi_{12}|, |\xi_{14}| \ll |\xi_1|$. We write the left-hand side as

$$|f(\xi_1) - f(\xi_1 - \xi_{12}) - f(\xi_1 - \xi_{14}) + f(\xi_1 - \xi_{12} - \xi_{14})|$$

which we can write as

$$|\int_0^1 \int_0^1 (\xi_{12} \cdot \nabla)(\xi_{14} \cdot \nabla) f(\xi_1 - s\xi_{12} - t\xi_{14}) \ ds dt|.$$

Since $\nabla^2 f(\xi_1 - s\xi_{12} - t\xi_{14})| = O(m_1^2)$, the claim follows.

From this lemma and (3.6), (3.7), (3.3), we obtain the following useful pointwise bound:

Corollary 4.2. For any $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4$, we have

$$|\tilde{\sigma}_4| \lesssim \frac{\min(m_1, m_2, m_3, m_4)^2}{\theta_0}.$$

Since, for $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4$, we have

$$|\sigma_4| \sim m_1 m_2 m_3 m_4 \lesssim \min(m_1, m_2, m_3, m_4)^2$$

we reduce matters to showing that

$$\int_{\Sigma_4: \max_{1 \le j \le 4} |\xi_j| > N} \min(m_1, m_2, m_3, m_4)^2 \prod_{j=1}^4 \frac{|\hat{u}_j(\xi_j)|}{m_j \langle \xi_j \rangle} \lesssim N^{-2} \prod_{j=1}^4 ||u_j||_{L^2(\mathbb{R}^2)}.$$

Note that at least two of $|\xi_1|$, $|\xi_2|$, $|\xi_3|$, $|\xi_4|$ need to be greater than or comparable to N. Without loss of generality we may assume that $|\xi_1|$, $|\xi_2| \gtrsim N$. Then $m_j \langle \xi_j \rangle \gtrsim$

 $N^{1-}\langle \xi_j \rangle^{0+}$ for j=1,2, while $\min(m_1,m_2,m_3,m_4)^2 \lesssim m_3 m_4$, so we reduce to showing that

$$\int_{\Sigma_4} \langle \xi_1 \rangle^{0-} \langle \xi_2 \rangle^{0-} \langle \xi_3 \rangle^{-1} \langle \xi_4 \rangle^{-1} \prod_{j=1}^4 |\hat{u}_j(\xi_j)| \lesssim \prod_{j=1}^4 \|u_j\|_{L^2(\mathbb{R}^2)},$$

which by Plancherel is equivalent to the estimate

$$\left| \int_{\mathbb{R}^2} v_1 v_2 v_3 v_4 \ dx \right| \lesssim \|v_1\|_{H^{0+}(\mathbb{R}^2)} \|v_2\|_{H^{0+}(\mathbb{R}^2)} \|v_3\|_{H^1(\mathbb{R}^2)} \|v_4\|_{H^1(\mathbb{R}^2)}$$

for some v_1, v_2, v_3, v_4 . But this easily follows from Sobolev embedding and Hölder. This proves Proposition 3.2. Note that this proof only required s > 0.

5. Modified local well-posedness

It remains to prove Proposition 3.3. From the hypotheses on u_0 we have

$$||Iu_0||_{H^1_{\sigma}(\mathbb{R}^2)} \lesssim A.$$

In order to use this bound, we need some spacetime estimates on the solution u. We recall the standard $X^{s,b}(\mathbb{R} \times \mathbb{R}^2)$ spaces for $s,b \in \mathbb{R}$, defined on spacetime Schwartz functions by the norm⁴

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^2)}:=\|\langle\xi\rangle^s\langle\tau+|\xi|^2\rangle^b\tilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R}^2)},$$

where

$$\tilde{u}(\tau,\xi) := \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-i(t\tau + x \cdot \xi)} u(t,x) \ dxdt$$

is the spacetime Fourier transform of u, and then for any time interval J, define the restricted norm $X^{s,b}(J \times \mathbb{R}^2)$ by the formula

$$||u||_{X^{s,b}(J\times\mathbb{R}^2)} := \inf\{||v||_{X^{s,b}(\mathbb{R}\times\mathbb{R}^2)} : v|_{J\times\mathbb{R}^2} = u\}$$

where v ranges over all functions in $X^{s,b}(\mathbb{R} \times \mathbb{R}^2)$ which agree with u on $J \times \mathbb{R}^2$. We caution that u and \overline{u} need not have comparable $X^{s,b}$ norms; this will complicate our notation a little bit but will not significantly affect the analysis.

We now fix an exponent b close to 1/2 (e.g. b := 0.6).

Proposition 5.1 (Modified local existence). Let u_0 be such that $||Iu_0||_{H^1_x(\mathbb{R}^2)} \lesssim A$, and u is a smooth-in-time, Schwartz-in-space solution to (1.1) on a time interval $[0, t_0]$, then if t_0 is sufficiently small depending on A, we have

$$||Iu||_{X^{1,b}([0,t_0]\times\mathbb{R}^2)} \lesssim A.$$

Proof. See [17, Proposition 3.2]. The proposition there was stated only for s > 4/7 and for an unspecified b, but it is not difficult to see that the argument in fact works for b := 0.6 and for all s > 1/2. (In fact, the argument works for all s > 0, though as s approaches 0 one needs to let b approach 1/2.)

In view of this proposition, we see that to prove Proposition 3.3 it suffices to prove the following estimates.

⁴Note that our sign conventions for the Schrödinger equation and the spacetime Fourier transform force the dispersion relation to be $\tau = -|\xi|^2$ rather than $\tau = +|\xi|^2$. Of course, these sign conventions are not crucial to our final results.

Proposition 5.2 (Quadrilinear estimate). For any $Iu \in X^{1,b}(\mathbb{R} \times \mathbb{R}^2)$ and $0 < t_0 < 1$ we have

(5.1)
$$\left| \int_{0}^{t_0} \Lambda_4([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha_4; u(t)) \right| dt \leq \left[N^{-2+} + \theta_0^{1/2} N^{-3/2+} \right] \|Iu\|_{X^{1,b}(\mathbb{R} \times \mathbb{R}^2)}^4.$$

Proposition 5.3 (Sextilinear estimate). For any $Iu \in X^{1,b}(\mathbb{R} \times \mathbb{R}^2)$ and $0 < t_0 < 1$ we have

(5.2)
$$|\int_0^{t_0} \Lambda_6([4iX(\tilde{\sigma}_4)]_{\text{sym}}; u(t)) \ dt| \lesssim \theta_0^{-1} N^{-3+} ||Iu||_{X^{1,b}(\mathbb{R} \times \mathbb{R}^2)}^6.$$

Remark 5.4. Observe that decreasing the threshold θ_0 between resonance and non-resonance improves the quadrilinear estimate (fewer resonant interactions) at the expense of the sextilinear estimate (more non-resonant interactions). The case $\theta = 1$ is essentially the case considered in [17].

The proof of these propositions will occupy the remainder of the paper.

Henceforth all spacetime norms will be on the full spacetime domain $\mathbb{R} \times \mathbb{R}^2$, and we shall omit this domain from the notation for brevity.

6.
$$X^{s,b}$$
 estimates

In this section we record some standard estimates involving the $X^{s,b}$ spaces which we will need in the sequel.

Let us say that a function u has spatial frequency N if its Fourier transform (either spatial or spacetime) is supported on the annulus $\{\langle \xi \rangle \sim N\}$. From the standard energy estimate $\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{0,1/2+}}$ (see e.g. [27, Corollary 2.10]) and Bernstein's inequality we have

Lemma 6.1 (Linear estimate). If u has spatial frequency N, then

(6.1)
$$||u||_{L^{\infty}_{t,x}} \lesssim N^{1} ||u||_{X^{0,1/2+}}.$$

We also have some standard bilinear estimates:

Lemma 6.2 (Bilinear Strichartz estimate). If u_1 , u_2 have spatial frequency N_1 , N_2 respectively, then

(6.2)
$$||u_1 u_2||_{L^2_{t,x}} \lesssim \frac{N_2^{1/2}}{N_1^{1/2}} ||u_1||_{X^{0,1/2+}} ||u_2||_{X^{0,1/2+}}.$$

If, furthermore, $N_2 \ll N_1$ and u_2 has Fourier support supported in a ball⁵ $\{\xi = \xi_0 + O(\theta N_2)\}$ of radius $O(\theta N_2)$ for some $0 < \theta < 1$, then we can improve the above estimate to

(6.3)
$$\|u_1 u_2\|_{L^2_{t,x}} \lesssim \frac{\theta^{1/2} N_2^{1/2}}{N_1^{1/2}} \|u_1\|_{X^{0,1/2+}} \|u_2\|_{X^{0,1/2+}}.$$

Proof. The estimate (6.2) is standard, see e.g. [5], [6] (see also [19, Lemma 3.4]). The second claim then follows by a Galilean transformation argument, shifting the frequencies of u_1, u_2 by about N_2 to ensure that u_2 now has frequency $\sim \theta N_2$ rather than N_2 , without significantly affecting the frequency of N_1 . Note that Galilean transforms do not affect the $X^{0,1/2+}$ norm or the $L^2_{t,x}$ norm of u_1u_2 .

 $^{^5 \}mathrm{This}$ ball is actually a cylinder if one also considers the time-frequency variable $\tau.$

Of course, it is advantageous to apply this estimate when $N_2 \leq N_1$ rather than when $N_1 \leq N_2$. We also make the trivial remark that we can replace u_1u_2 by $\overline{u_1}u_2$, $u_1\overline{u_2}$, or $\overline{u_1u_2}$ without affecting the estimate.

7. Proof of Sextilinear estimate

In this section we prove Proposition 5.3, which is the easier of the two propositions, as it does not require any fine control on the resonant interactions⁶. The left-hand side of (5.2) can be expanded as

$$|\int_0^{t_0} \left[\int_{\Sigma_6} [4iX(\tilde{\sigma}_4)]_{\text{sym}}(\xi)\hat{u}(t,\xi_1)\dots\hat{\overline{u}}(t,\xi_6)\right] dt|.$$

If $\max(|\xi_1|, \ldots, |\xi_6|) < N/3$, then (by (3.5), (3.6)), $4iX(\tilde{\sigma}_4) = 1$, and thus $[4iX(\tilde{\sigma}_4)]_{\text{sym}}$ vanishes. We can thus restrict to the region $\max(\xi_1, \ldots, \xi_6) \ge N/3$. We then remove the symmetry and reduce to showing that

$$\left| \int_0^{t_0} \left[\int_{\Sigma_6: \max(|\xi_1|, \dots, |\xi_6|) \ge N/3} X(\tilde{\sigma}_4) \hat{u}(t, \xi_1) \dots \hat{\overline{u}}(t, \xi_6) \right] dt \right| \lesssim \theta_0^{-1} N^{-3+} ||Iu||_{X^{1,b}}^6.$$

Because the $X^{s,b}$ norm uses the spacetime Fourier transform, we will be forced for technical reasons⁷ to write the left-hand side in terms of the spacetime Fourier transform. Indeed, this left-hand side becomes

$$|\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Sigma_{6}} \hat{1}_{[0,t_{0}]}(\tau_{0}) 1_{\max(|\xi_{1}|,\dots,|\xi_{6}|) \geq N/3} X(\tilde{\sigma}_{4}) \tilde{u}(\tau_{1},\xi_{1}) \dots \tilde{\overline{u}}(\tau_{6},\xi_{6}) \right] d\tau_{1} \dots d\tau_{6}|$$

where $\tau_0 := -\tau_1 - \ldots - \tau_6$. Using Corollary 4.2 and the bound

(7.1)
$$\hat{1}_{[0,t_0]}(\tau_0) = \langle \tau_0 \rangle^{-1}$$

we can estimate this quantity by

$$\lesssim \frac{1}{\theta_0} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Sigma_6} \langle \tau_0 \rangle^{-1} 1_{\max(|\xi_1|, \dots, |\xi_6|) \geq N/3} \min(m_{123}, m_4, m_5, m_6)^2 | \tilde{u}(\tau_1, \xi_1) | \dots | \tilde{\overline{u}}(\tau_6, \xi_6) | \right] d\tau_1 \dots d\tau_6$$

where $m_{123} = m(\xi_{123})$. It will be convenient to hide the conjugations by using the norm

$$||u||_{\tilde{X}^{s,b}} := \inf\{||u_1||_{X^{s,b}} + ||\overline{u_2}||_{X^{s,b}} : u = u_1 + u_2\}$$

so it suffices to show the estimate

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Sigma_6} \langle \tau_0 \rangle^{-1} 1_{\max(|\xi_1|, \dots, |\xi_6|) \ge N/3} \min(m_{123}, m_4, m_5, m_6)^2 \prod_{j=1}^6 |\tilde{u}_j(\tau_j, \xi_j)| \right] d\tau_1 \dots d\tau_6$$

$$\lesssim N^{-3+} \prod_{j=1}^{6} ||Iu_j||_{\tilde{X}^{1,b}}.$$

If we let $|\xi_1^*| \ge ... \ge |\xi_6^*|$ be the six magnitudes $|\xi_1|,...,|\xi_6|$ in order, we observe that $\min(m_{123},m_4,m_5,m_6)^2 \le m(\xi_4^*)^2$. Inserting this bound, the left-hand side is

⁶Indeed, there seems to be a general principle when applying the *I*-method that terms which are more multilinear (and hence have fewer derivatives) are easier to estimate than terms which are less multilinear. This phenomenon, which is related to the sub-criticality of the regularities being considered, explains why it is beneficial to adjust the modified energy using correction terms, as this makes the error terms more multilinear.

⁷The specific issue is that we cannot automatically reduce to the case where the spatial Fourier transforms of the u_j are non-negative. In previous literature this difficulty was avoided by using the Coifman-Meyer multiplier theorem, but the symbol here does not obey Coifman-Meyer type estimates and so this theorem is not applicable.

now symmetric in ξ_1, \ldots, ξ_6 , so we can assume that $|\xi_1| \ge \ldots \ge |\xi_6|$. The constraint $\max(|\xi_1|, \ldots, |\xi_6|) \ge N/3$ then implies $|\xi_2| \gtrsim N$. We thus need to show that

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Sigma_{6}} \langle \tau_{0} \rangle^{-1} 1_{|\xi_{1}| \geq \dots \geq |\xi_{6}|} 1_{|\xi_{2}| \gtrsim N} m(\xi_{4})^{2} \prod_{j=1}^{6} |\tilde{u}_{j}(\tau_{j}, \xi_{j})| \right] d\tau_{1} \dots d\tau_{6} \lesssim N^{-3+} \prod_{j=1}^{6} ||Iu_{j}||_{\tilde{X}^{1,b}}.$$

Partitioning up into Littlewood-Paley pieces, it suffices to show that

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Sigma_{6}} \langle \tau_{0} \rangle^{-1} \prod_{j=1}^{6} |\tilde{u}_{j}(\tau_{j}, \xi_{j})| \right] d\tau_{1} \dots d\tau_{6} \lesssim \frac{N^{-3+} N_{1}^{0-} \prod_{j=1}^{6} m(N_{j}) N_{j} \|u_{j}\|_{\tilde{X}^{0,b}}}{m(N_{4})^{2}}$$

whenever $N_1 \gtrsim N_2 \gtrsim \ldots \gtrsim N_6 \gtrsim 1$, and each u_j has spatial frequency N_j . Note we may assume $N_1 \sim N_2$ since the Σ_6 integral vanishes otherwise. Since the definition of $\tilde{X}^{1,b}$ uses only the magnitude of the spacetime Fourier transform, we may take all of the \tilde{u}_j to be non-negative and thus omit the absolute value signs. The left-hand side can now be written using spacetime convolutions as

(7.2)
$$\int_{\mathbb{R}} \langle \tau \rangle^{-1} \tilde{u}_1 * \dots * \tilde{u}_6(\tau, 0) \ d\tau.$$

It is slightly unfortunate that $\langle \tau \rangle^{-1}$ barely fails to be integrable. However, if we introduce the logarithmic weight $w(\tau) := 1 + \log^2 \langle \tau \rangle$, then $\langle \tau \rangle^{-1} w^{-1}$ is integrable. Also, from the elementary estimate $w(\tau_1 + \ldots + \tau_6) \lesssim w(\tau_1) \ldots w(\tau_6)$ we have the pointwise bound

$$\tilde{u}_1 * \ldots * \tilde{u}_6 \lesssim w^{-1}[(w\tilde{u}_1) * \ldots (w\tilde{u}_6)]$$

and thus we can bound

$$(7.2) \lesssim \|(w\tilde{u}_1) * \ldots * (w\tilde{u}_6)\|_{L^{\infty}}$$

Thus it will suffice to show that

$$\|(w\tilde{u}_1)*...*(w\tilde{u}_6)\|_{L^{\infty}_{\tau,\xi}} \lesssim \frac{N^{-3+}N_1^{0-}\prod_{j=1}^6 m(N_j)N_j\|u_j\|_{\tilde{X}^{0,b}}}{m(N_4)^2}$$

If v_j denotes the function with spacetime Fourier transform $\tilde{v}_j = w\tilde{u}_j$, one easily verifies that

$$||v_j||_{\tilde{X}^{0,b-}} \lesssim \log^2(1+N_1)||u_j||_{\tilde{X}^{0,b}}$$

and so it will suffice to show that

$$\|\tilde{v}_1 * \dots * \tilde{v}_6\|_{L^{\infty}_{\tau,\xi}} \lesssim \frac{N^{-3+}N_1^{0-} \prod_{j=1}^6 m(N_j)N_j\|v_j\|_{\tilde{X}^{0,b-}}}{m(N_4)^2}$$

By Hausdorff-Young it suffices to show that

$$||v_1 \dots v_6||_{L^1_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \lesssim \frac{N^{-3+} N_1^{0-} \prod_{j=1}^6 m(N_j) N_j ||v_j||_{\tilde{X}^{0,b-}}}{m(N_4)^2}.$$

Since the left-hand side is insensitive to conjugation, it suffices to show that

$$||v_1 \dots v_6||_{L^1_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \lesssim \frac{N^{-3+} N_1^{0-} \prod_{j=1}^6 m(N_j) N_j ||v_j||_{X^{0,b-}(\mathbb{R} \times \mathbb{R}^2)}}{m(N_4)^2}.$$

Estimating v_1v_3 and v_2v_4 in $L^2_{t,x}$ using (6.2), and v_5 , v_6 in $L^{\infty}_{t,x}$ using (6.1), and applying Hölder's inequality, we reduce to showing that

$$\frac{N_3^{1/2}}{N_1^{1/2}} \frac{N_4^{1/2}}{N_2^{1/2}} N_5^1 N_6^1 \lesssim \frac{N^{-3+} N_1^{0-} \prod_{j=1}^6 m(N_j) N_j}{m(N_4)^2}.$$

We rearrange (using $N_1 \sim N_2$) as

$$1 \lesssim N^{-3+} m(N_1)^2 N_1^{3-} m(N_3) N_3^{1/2} m(N_4) N_4^{1/2} m(N_4)^{-2} m(N_5) m(N_6).$$

Since the function $m(|\xi|)$ is non-increasing in $|\xi|$, we may reduce to the case $N_4 = N_5 = N_6$, which becomes

$$1 \lesssim N^{-3+} m(N_1)^2 N_1^{3-} m(N_3) N_3^{1/2} m(N_4) N_4^{1/2}.$$

This is true since $m(N_3)N_3^{1/2} \ge m(N_4)N_4^{1/2} \ge 1$ and $m(N_1)N_1 \gtrsim N$. This concludes the proof of Proposition 5.3.

8. Proof of quadrilinear estimate

We now begin the proof of Proposition 5.2. We shall begin by mimicking the proof of Proposition 5.3, but we will find that there are a few cases, particularly those involving resonant interactions, which require more careful attention, both in the pointwise estimates on the multiplier, and on the bilinear estimates needed to handle the final expression.

From (3.6) we have

$$([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha_4)(\xi) = [-2iX(\sigma_2)]_{\text{sym}}1_{\Omega_r}$$

where the resonant set

$$\Omega_r := \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 : \max(|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4|) > N; |\cos \angle (\xi_{12}, \xi_{14})| < \theta_0 \}$$

is the complement of Ω_{nr} . Thus the left-hand side of (5.1) can be expressed using spacetime Fourier transforms similarly to the previous section as

$$\left| \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \widehat{\mathbf{1}_{[0,t_0]}}(\tau_0) \left[\int_{\Omega_r} [-2iX(\sigma_2)]_{\text{sym}}(\xi) \tilde{u}(\tau_1,\xi_1) \dots \tilde{\overline{u}}(\tau_4,\xi_4) \right] d\tau_1 \dots d\tau_4 \right|$$

where $\tau_0 := -\tau_1 - \ldots - \tau_4$. Using the bound (7.1), we can bound this by

$$\lesssim \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \langle \tau_0 \rangle^{-1} \left[\int_{\Omega_-} |[X(\sigma_2)]_{\text{sym}}||\tilde{u}(\tau_1, \xi_1)| \dots |\tilde{\overline{u}}(\tau_4, \xi_4)|\right] d\tau_1 \dots d\tau_4.$$

By symmetry we may reduce to the region where $|\xi_1| \ge |\xi_2|$, $|\xi_3|$, $|\xi_4|$ and $|\xi_2| \ge |\xi_4|$, or equivalently we may replace Ω_r with

$$\Omega'_r := \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4 : |\xi_1| \ge |\xi_2|, |\xi_3|, |\xi_4|; |\xi_2| \ge |\xi_4|; |\xi_1| > N; |\cos \angle(\xi_{12}, \xi_{14})| < \theta_0\}.$$

It thus suffices as in the previous section to show the estimate

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \langle \tau_0 \rangle^{-1} \left[\int_{\Omega'_r} |[X(\sigma_2)]_{\text{sym}}|| \prod_{j=1}^4 \tilde{u}_j^*(\tau_j, \xi_j) \right] d\tau_1 \dots d\tau_4$$

$$\lesssim N_1^{0-} [N^{-2+} + \theta_0^{1/2} N^{-3/2+}] \prod_{j=1}^4 m(N_j) N_j ||u_j||_{\tilde{X}^{0,b}(\mathbb{R} \times \mathbb{R}^2)}$$

whenever

$$N_1 \gtrsim N, N_2, N_3, N_4 \gtrsim 1; \quad N_2 \gtrsim N_4,$$

and u_1, \ldots, u_4 have spatial frequency N_1, \ldots, N_4 respectively, and have non-negative spacetime Fourier transform. Here we adopt the convention that $u_j^* = u_j$ when j is odd and $u_j^* = \overline{u_j}$ when j is even. Note that we may also assume that

$$N_1 \sim N_2$$

for if $N_2 \ll N_1$ then ξ_{12}, ξ_{14} make a small angle with ξ_1 and it will be impossible to satisfy the condition $|\cos \angle (\xi_{12}, \xi_{14})| \leq \theta_0$. The constraints on N_1, N_2, N_3, N_4 are now symmetric under swapping N_1, N_3 with N_2, N_4 respectively and so we may now also assume $N_3 \gtrsim N_4$. To summarize, the frequencies N_1, N_2, N_3, N_4 are now known to obey the relations

(8.1)
$$N_1 \sim N_2 \gtrsim N; \quad N_2 \gtrsim N_3 \gtrsim N_4 \gtrsim 1.$$

Using the weight w and the functions v_j as in the previous section, we reduce to showing that

(8.2)
$$\sup_{\tau} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Omega_{r}'} |[X(\sigma_{2})]_{\text{sym}}| \prod_{j=1}^{4} \tilde{v}_{j}^{*}(\tau_{j}, \xi_{j})] \delta(\tau_{0} - \tau) d\tau_{1} \dots d\tau_{4} \right] \\ \lesssim N_{1}^{0-} [N^{-2+} + \theta_{0}^{1/2} N^{-3/2+}] \prod_{j=1}^{4} m(N_{j}) N_{j} ||v_{j}||_{\tilde{X}^{0,b-}}.$$

We now dispose of an easy case, in which we will not use the $\theta_0^{1/2}N^{-3/2+}$ term on the right-hand side. Suppose that

$$N_4 \gtrsim N$$
 or $N_3 \sim N_4$.

Using Lemma 4.1 we have

$$(8.3) [X(\sigma_2)]_{\text{sym}} \lesssim m(N_1)^2 |\xi_{14}| |\xi_{12}| \lesssim m(N_1)^2 N_1 N_3$$

since $|\xi_{12}| = |\xi_{34}|$ is bounded by N_3 . Gathering some terms and simplifying using (8.1), we reduce to showing that

$$\sup_{\tau} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left[\int_{\Omega'_{\tau}} \prod_{j=1}^{4} \tilde{v}_{j}^{*}(\tau_{j}, \xi_{j}) \right] \delta(\tau_{0} - \tau) d\tau_{1} \cdots d\tau_{4}$$

$$\lesssim N_{1}^{1-} \left[N^{-2+} + \theta_{0}^{1/2} N^{-3/2+} \right] m(N_{3}) m(N_{4}) N_{4} \prod_{j=1}^{4} \|v_{j}\|_{\tilde{X}^{0,b-}}.$$

We estimate the left-hand side by

$$\|\tilde{v}_1^**\ldots*\tilde{v}_4^*\|_{L^{\infty}_{\tau,\xi}}\lesssim \|v_1\overline{v_2}v_3\overline{v_4}\|_{L^1_{t,x}}.$$

Applying (6.2) to v_1v_3 and v_2v_4 , followed by Cauchy-Schwarz, we reduce to showing that

$$\frac{N_3^{1/2}}{N_1^{1/2}} \frac{N_4^{1/2}}{N_2^{1/2}} \lesssim N_1^{1-} N^{-2+} m(N_3) m(N_4) N_4$$

which simplifies using (8.1) to

$$1 \lesssim N^{-2+} N_1^{2-} m(N_3) N_3^{-1/2} m(N_4) N_4^{1/2}.$$

If $N_3 \sim N_4$, then this becomes

$$1 \lesssim N^{-2+} N_1^{2-} m(N_3)^2;$$

since $m(N_1)^2 \lesssim m(N_3)^2$, $m(\xi)^{2-}|\xi|^2$ is essentially increasing in $|\xi|$, and $N_1 \gtrsim N$, we conclude the claim. If instead $N_4 \gtrsim N$, we have $m(N_4)N_4^{1/2} \gtrsim N^{1/2}$ and $m(N_3)N_3^{-1/2} \gtrsim m(N_1)N_1^{-1/2}$, thus reducing to

$$1 \lesssim N^{-3/2+} N_1^{3/2-} m(N_1)$$

which follows since $N_1 \gtrsim N$.

To conclude Theorem 1.1 we only need to establish (8.2) in the remaining case when

(8.4)
$$N_1 \sim N_2 \gtrsim N, N_3 \gg N_4 \gtrsim 1.$$

It is here that we shall need to exploit the resonance constraint $|\cos \angle(\xi_{12}, \xi_{14})| \le \theta_0$ more fully. Specifically, we shall need the following improved bound on $|[X(\sigma_2)]_{\text{sym}}|$.

Lemma 8.1. Let $N_1, ..., N_4$ be as in (8.4), and let $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega'_r$ be such that $|\xi_j| \sim N_j$ for j = 1, 2, 3, 4. Then

$$|[X(\sigma_2)]_{\text{sym}}| \lesssim m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2$$

(compare with (8.3)).

Proof. The new idea is to exploit heavily the spherical symmetry of m. From (3.4) we have

$$|[X(\sigma_2)]_{\text{sym}}| \lesssim |m(|\xi_1|)^2 |\xi_1|^2 - m(|\xi_2|)^2 |\xi_2|^2 + m(N_3)^2 N_3^2 + m(N_4)^2 N_4^2$$

Since the function $x \mapsto m(x)^2 x^2$ is essentially increasing, it suffices to show that

$$m(|\xi_1|)^2 |\xi_1|^2 - m(|\xi_2|)^2 |\xi_2|^2 = m(N_1)^2 (O(N_3 N_1 \theta_0) + O(N_3)^2).$$

On the region $x \sim N_1$, the function $x \mapsto m(x)^2 x^2$ has derivative $O(m(N_1)^2 N_1)$. Thus we can bound the left-hand side by

$$m(N_1)^2 N_1 ||\xi_1| - |\xi_2|| \sim m(N_1)^2 ||\xi_1|^2 - |\xi_2|^2 ||\xi_1||^2 - |\xi_2|^2 ||\xi_1||^2 - |\xi_2||^2 ||\xi_1||^2 - |\xi_2||^2 ||\xi_1||^2 - ||\xi_2||^2 - ||\xi_1||^2 - ||\xi_2||^2 - ||\xi_1||^2 - ||\xi_2||^2 - ||\xi_2||$$

However, on Ω'_r we see from (3.3) that

$$-|\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 + |\xi_4|^2 = -2|\xi_{12}||\xi_{14}|\cos\angle(\xi_{12},\xi_{14}) = O(N_3N_1\theta_0)$$

and hence

$$|\xi_1|^2 - |\xi_2|^2 = O(N_3 N_1 \theta_0) + O(N_3)^2$$

and the claim follows.

We now replace the resonance constraint $|\cos \angle(\xi_{12}, \xi_{14})| \leq \theta_0$ with a simpler constraint. Observe from elementary trigonometry that

$$\angle(\xi_1, \xi_{14}) = O(\frac{N_4}{N_1}); \quad \angle(\xi_3, \xi_{34}) = O(\frac{N_4}{N_3})$$

and hence (since cosine is Lipschitz)

$$|\cos \angle(\xi_1, \xi_3)| = |\cos \angle(\xi_{14}, \xi_{34})| + O(\frac{N_4}{N_3}) = |\cos \angle(\xi_{14}, \xi_{12})| + O(\frac{N_4}{N_3}).$$

Thus on the resonance set Ω_r we have

$$|\cos \angle(\xi_1, \xi_3)| \lesssim \theta_0 + \frac{N_4}{N_3}.$$

From these observations, we can now bound the left-hand side of (8.2) by

$$[m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2] \|\tilde{v}_2^* * \tilde{v}_4^* * \tilde{F}\|_{L^{\infty}_{\tau,\xi}}$$

where

$$F(t,x) := \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1_{|\cos \angle(\xi_1,\xi_3)| \lesssim \theta_0 + \frac{N_4}{N_3}} e^{i(t(\tau_1 + \tau_3) + x \cdot (\xi_1 + \xi_3))} \tilde{v}_1(\tau_1,\xi_1) \tilde{v}_3(\tau_3,\xi_3) d\xi_1 d\xi_3 d\tau_1 d\tau_3.$$

Applying Cauchy-Schwarz and (6.2) we can thus estimate the left-hand side of (8.2) by

$$\lesssim [m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2] \frac{N_4^{1/2}}{N_2^{1/2}} \|v_2\|_{X^{0,b-}} \|v_4\|_{X^{0,b-}} \|F\|_{L^2_{t,x}}.$$

Inserting this into (8.2) and gathering terms using (8.4), we reduce to showing that (8.5)

$$||F||_{L^{2}_{t,x}} \lesssim \frac{m(N_{1})^{2} N_{1}^{5/2-} m(N_{3}) N_{4}^{1/2} [N^{-2+} + \theta_{0}^{1/2} N^{-3/2+}]}{m(N_{1})^{2} N_{1} \theta_{0} + m(N_{3})^{2} N_{3}} ||v_{1}||_{X^{0,b-}} ||v_{3}||_{X^{0,b-}}.$$

Note that we have dropped $m(N_4)$ here because our assumptions imply that $N_4 \lesssim N$ so $m(N_4) \sim 1$.

To proceed further we use

Lemma 8.2 (Angularly refined bilinear Strichartz estimate). Let $0 < N_1 \le N_2$ and $0 < \theta < \frac{1}{50}$. Then for any $v_1, v_2 \in X^{0,1/2+}$ with spatial frequencies N_1, N_2 respectively, the spacetime function

$$F(t,x) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(t(\tau_1 + \tau_2) + x \cdot (\xi_1 + \xi_2))} 1_{|\cos \angle(\xi_1, \xi_2)| \le \theta} \tilde{v}_1(\tau_1, \xi_1) \tilde{v}_2(\tau_2, \xi_2) \ d\xi_1 d\xi_2$$

obeys the bound

(8.6)
$$||F||_{L^{2}_{t,x}} \lesssim \theta^{1/2} ||v_{1}||_{X^{0,1/2+}} ||v_{2}||_{X^{0,1/2+}}.$$

Proof. We may assume that $\theta \ll N_1/N_2$ since the claim follows from (6.2) otherwise. By standard averaging arguments (see e.g. [27, Lemma 2.9]) it suffices to prove the claim for L^2 free solutions, or more precisely that if $\phi_1, \phi_2 \in L_x^2$ have spatial frequency N_1, N_2 respectively and

$$F(t,x) := \int_{\mathbb{D}^2} \int_{\mathbb{D}^2} 1_{|\cos \angle(\xi_1,\xi_2)| \le \theta} e^{-it(|\xi_1|^2 + \xi_2|^2)} e^{ix \cdot (\xi_1 + \xi_2)} \hat{\phi}_1(\xi_1) \hat{\phi}_2(\xi_2) \ d\xi_1 d\xi_2$$

then

(8.7)
$$||F||_{L^{2}_{t,x}} \lesssim \theta^{1/2} ||\phi_{1}||_{L^{2}_{x}} ||\phi_{2}||_{L^{2}_{x}}.$$

We first verify the estimate in the special case when the Fourier transform of ϕ_j is supported in an angular sector $\{\xi_j: \arg(\xi_j) = l_j\theta + O(\theta)\}$ of width $O(\theta)$ for j=1,2, where l_1,l_2 are arbitrary integers $1 \leq l_1,l_2 \leq \frac{2\pi}{\theta}$. Observe that the spacetime Fourier transform of F is given by the formula

$$\tilde{F}(\tau,\xi) = \int_{\mathbb{R}^2} 1_{|\xi_1| \sim N_1} 1_{|\xi - \xi_1| \sim N_2} 1_{\arg(\xi_1) = l_1 \theta + O(\theta)} 1_{\arg(\xi - \xi_1) = l_2 \theta + O(\theta)} 1_{|\cos \angle(\xi_1, \xi - \xi_1)| \le \theta}
\hat{\phi}_1(\xi_1) \hat{\phi}_2(\xi - \xi_1) \delta(|\xi_1|^2 + |\xi - \xi_1|^2 + \tau) d\xi_1.$$

From the cosine rule

$$|\xi|^2 = |\xi_1|^2 + |\xi - \xi_1|^2 + 2|\xi_1||\xi - \xi_1|\cos\angle(\xi_1, \xi - \xi_1)$$

and the hypothesis $N_1 \leq N_2$ we thus see that $\tilde{F}(\tau,\xi)$ is zero unless $|\xi| \sim N_2$, and $0.9|\xi|^2 \leq -\tau \leq 1.1|\xi|^2$. Thus we may take absolute values followed by Cauchy-Schwarz and estimate

$$\begin{split} |\tilde{F}(\tau,\xi)|^2 &\leq 1_{|\xi| \sim N_2, 0.9|\xi|^2 \leq -\tau \leq 1.1|\xi|^2} \left[\int_{\mathbb{R}^2} 1_{|\xi_1| \sim N_1} 1_{\arg(\xi_1) = l_1 \theta + O(\theta)} 1_{\arg(\xi - \xi_1) = l_2 \theta + O(\theta)} \right] \\ & |\hat{\phi}_1(\xi_1)| |\hat{\phi}_2(\xi - \xi_1)| \delta(|\xi_1|^2 + |\xi - \xi_1|^2 + \tau) \ d\xi_1 \end{split}^2 \\ &\leq \int_{\mathbb{R}^2} |\hat{\phi}_1(\xi_1)|^2 |\hat{\phi}_2(\xi - \xi_1)|^2 \delta(|\xi_1|^2 + |\xi - \xi_1|^2 + \tau) \ d\xi_1 \\ & \times \sup_{|\xi| \sim N_2, 0.9|\xi|^2 \leq -\tau \leq 1.1|\xi|^2} \int_{\mathbb{R}^2} 1_{|\xi_1| \sim N_1} 1_{\arg(\xi_1) = l_1 \theta + O(\theta)} 1_{\arg(\xi - \xi_1) = l_2 \theta + O(\theta)} \\ & \delta(|\xi_1|^2 + |\xi - \xi_1|^2 + \tau) \ d\xi_1. \end{split}$$

Integrating this in τ and ξ and using Plancherel's theorem, we see that to prove (8.7) it will suffice to show that

$$\int_{\mathbb{R}^2} 1_{|\xi_1| \sim N_1} 1_{\arg(\xi_1) = l_1 \theta + O(\theta)} 1_{\arg(\xi - \xi_1) = l_2 \theta + O(\theta)} \delta(|\xi_1|^2 + |\xi - \xi_1|^2 + \tau) \ d\xi_1 \lesssim \theta$$

or equivalently that

$$|\{\xi_1 \in \mathbb{R}^2 : \arg(\xi_1) = l_1\theta + O(\theta); \arg(\xi - \xi_1) = l_2\theta + O(\theta); |\xi_1 - \xi/2| = r + O(\varepsilon/N_2)\}| \lesssim \theta\varepsilon$$

whenever $|\xi| \sim N_2$ and $r \sim N_2$, and ε is sufficiently small. But if ξ_1 is closer to 0 than to ξ , the angular constraint $\arg(\xi - \xi_1) = l_2\theta + O(\theta)$ restricts the circle $|\xi_1 - \xi/2| = r$ to an arc of length $O(\theta)$; similarly if ξ_1 is closer to ξ than to 0 using the angular constraint $\arg(\xi_1) = l_1\theta + O(\theta)$. The claim follows.

Now we establish the general case. We can subdivide

$$\phi_1 = \sum_{l_1} \phi_{1,l_1}; \quad \sum_{l_2} \phi_{1,l_2}$$

where l_1, l_2 range over the integers between 0 and $2\pi/\theta$, and the Fourier transform of ϕ_{j,l_j} is supported in an angular sector $\{\xi_j : \arg(\xi_j) = l_j\theta + O(\theta)\}$. Then we have

$$F(t,x) = \sum_{l_1,l_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1_{|\xi_1| \sim N_1} 1_{|\xi_2| \sim N_2} 1_{|\cos \angle(\xi_1,\xi_2)| \le \theta} e^{ix \cdot (\xi_1 + \xi_2)} e^{-it|\xi_1|^2}$$
$$\hat{\phi}_{1,l_1}(\xi_1) e^{-it|\xi_2|^2} \hat{\phi}_{2,l_2}(\xi_2) \ d\xi_1 d\xi_2.$$

But observe that if $|\cos \angle(\xi_1, \xi_2)| \le \theta$ then $|\arg(\xi_1) - \arg(\xi_2)| = \pi/2 + O(\theta)$ or $|\arg(\xi_1) - \arg(\xi_2)| = 3\pi/2 + O(\theta)$ and hence $|l_2 - l_1| = \pi/(2\theta) + O(1)$ or $|l_2 - l_1| = 3\pi/(2\theta) + O(1)$. From this, the triangle inequality, and the preceding computation we have

$$\|F\|_{L^2_{t,x}} \lesssim \sum_{l_1,l_2: |l_2-l_1| = \pi/(2\theta) + O(1) \text{ or } 3\pi/2\theta + O(1)} \theta^{1/2} \|\phi_{1,l_1}\|_{L^2_x} \|\phi_{2,l_2}\|_{L^2_x}.$$

Observe that for each l_1 there are only O(1) values of l_2 which contribute to this sum, and vice versa. Thus by Schur's test

$$\|F\|_{L^2_{t,x}} \lesssim \theta^{1/2} (\sum_{l_1} \|\phi_{1,l_1}\|_{L^2_x}^2)^{1/2} (\sum_{l_2} \|\phi_{2,l_2}\|_{L^2_x}^2)^{1/2}$$

and the claim follows by Plancherel's theorem.

Remark 8.3. In the regime $\theta \ll N_1/N_2$, the estimate is sharp, as can be seen by using (time-localized) free solutions of data ϕ_1, ϕ_2 whose Fourier transforms are indicator functions of the rectangles $[N_1 - \theta N_2, N_1 + \theta N_2] \times [-\theta N_1, \theta N_1]$ and $[-\theta N_2, \theta N_2] \times [N_2 - \theta N_1, N_2 + \theta N_1]$ respectively; we omit the details. Of course in the regime $\theta \gg N_1/N_2$ the estimate (6.2) is superior. It appears that similar estimates also hold if the angular constraint $|\cos \angle(\xi_1, \xi_2)| \le \theta$ is replaced with similar constraints such as $|\angle(\xi_1,\xi_2) - \alpha| \le \theta$ for some $\alpha \gg \theta$, but we will not need such variants here.

Applying this lemma, we reduce to showing that

$$(8.8) (\theta_0 + \frac{N_4}{N_3})^{1/2} \lesssim \frac{m(N_1)^2 N_1^{5/2} - m(N_3) N_4^{1/2} [N^{-2+} + \theta_0^{1/2} N^{-3/2+}]}{m(N_1)^2 N_1 \theta_0 + m(N_3)^2 N_3}$$

We establish (8.8) by splitting into some cases.

First, suppose that $N_3 \gtrsim \frac{N_4}{\theta_0}$, then the left-hand side of (8.8) can be bounded by $O(\theta_0^{1/2})$. If we bound the denominator on the right-hand side crudely from above by $O(m(N_1)^2N_1)$, and discard the N^{-2+} term in the numerator, we thus reduce to showing that

$$\theta_0^{1/2} \lesssim \frac{m(N_1)^2 N_1^{5/2} m(N_3) N_4^{1/2} \theta_0^{1/2} N^{-3/2}}{m(N_1)^2 N_1}$$

which simplifies to

$$1 \lesssim N_1^{3/2-} m(N_3) N_4^{1/2} N^{-3/2+}$$
.

But this is clear by estimating $m(N_3) \gtrsim m(N_1)$, $N_4^{1/2} \gtrsim 1$ and using $N_1 \gtrsim N$. Henceforth we assume $N_3 \ll \frac{N_4}{\theta_0}$, so that the left-hand side of (8.8) is $N_4^{1/2} N_3^{-1/2}$, which allows us to rearrange (8.8) as

$$(8.9) \ m(N_1)^2 N_1 \theta_0 + m(N_3)^2 N_3 \lesssim m(N_1)^2 N_1^{5/2} - m(N_3) N_3^{1/2} [N^{-2+} + \theta_0^{1/2} N^{-3/2+}].$$

Let us first consider the bound for $m(N_1)^2N_1\theta_0$. Discarding the N^{-2+} factor on the right, we reduce to

$$m(N_1)^2 N_1 \theta_0 \lesssim m(N_1)^2 N_1^{5/2} m(N_3) N_3^{1/2} \theta_0^{1/2} N^{-3/2}$$

which simplifies to

$$\theta_0^{1/2} \lesssim N_1^{3/2-} m(N_3) N_3^{1/2} N^{-3/2+}$$

But this is certainly true since $m(N_3)N_3^{1/2} \gtrsim 1$ and $\theta_0 \leq 1$.

Now we consider the bound for $m(N_3)^2N_3$ in (8.9). Discarding the $\theta_0^{1/2}N^{-3/2+}$ term, we reduce to

$$m(N_3)^2 N_3 \lesssim m(N_1)^2 N_1^{5/2-} m(N_3) N_3^{1/2} N^{-2+}$$

which simplifies to

$$m(N_3)N_3^{1/2} \lesssim m(N_1)^2 N_1^{5/2-} N^{-2+}$$
.

Since $m(N_3)N_3^{1/2} \lesssim m(N_1)N_1^{1/2}$, we reduce to

$$1 \lesssim m(N_1)N_1^{2-}N^{-2+}$$

which is true since $N_1 \gtrsim N$. This completes the proof of Proposition 5.2, and Theorem 1.1 follows.

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